Problem 1.44

Verify by direct substitution that the function $\phi(t) = A\sin(\omega t) + B\cos(\omega t)$ of (1.56) is a solution of the second-order differential equation (1.55), $\ddot{\phi} = -\omega^2\phi$. (Since this solution involves two arbitrary constants—the coefficients of the sine and cosine functions—it is in fact the general solution.)

Solution

Plug the given formula for $\phi(t)$ into the differential equation.

$$\ddot{\phi} \stackrel{?}{=} -\omega^2\phi$$

$$\frac{d^2}{dt^2}[A\sin(\omega t) + B\cos(\omega t)] \stackrel{?}{=} -\omega^2[A\sin(\omega t) + B\cos(\omega t)]$$

$$\frac{d}{dt}[A\omega\cos(\omega t) - B\omega\sin(\omega t)] \stackrel{?}{=} -\omega^2[A\sin(\omega t) + B\cos(\omega t)]$$

$$[-A\omega^2\sin(\omega t) - B\omega^2\cos(\omega t)] \stackrel{?}{=} -\omega^2[A\sin(\omega t) + B\cos(\omega t)]$$

$$-\omega^2[A\sin(\omega t) + B\cos(\omega t)] = -\omega^2[A\sin(\omega t) + B\cos(\omega t)]$$

This is a true statement, so $\phi(t) = A\sin(\omega t) + B\cos(\omega t)$ is indeed a solution to $\ddot{\phi} = -\omega^2\phi$.